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A HYBRID DISPLACEMENT BOUNDARY ELEMENT FORMULATION FOR REISSNER PLATE WITH QUADRATIC ELEMENTS

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Abstract. This paper presents the derivation of a new boundary element formulation for plate bending problems. The Reissner's plate bending theory is employed. Unlike the conventional direct or indirect formulations, the proposed integral equation is based on minimizing the relevant energy functional. In doing so, variational methods are used. A collocation based series, similar to the one used in the indirect discrete boundary element method, is used to remove domain integrals. Hence, a fully boundary integral equation is formulated. The main advantage of the proposed formulation is production of a symmetric stiffness matrix similar to that obtained in the finite element method. Numerical examples are presented to demonstrate the accuracy and the validity of the proposed formulation.

1. INTRODUCTION

Concerning the application of the BEM for thick plate bending problems, the direct boundary element formulation originally developed by Vander Weeën [1]. Hence, several applications were considered based on this theory; for example: Barcellos and Silva [2] extended the formulation to Mindlin plates. El-Zafrany et al. [3] divided the formulation into kernels for thin and others for thick plates. Ribeiro and Venturini [4] discussed the application of elasto-plastic analysis to the direct formulation. Westphal et al. [5] studied the fundamental solution used in plates. Marczak and Creus [6] considered the evaluation of singular integrals in the direct integral equation formulation. Fernandes and Konda [7] coupled the formulation with beams. To the author's best knowledge, none of these formulations considered a variational boundary integral formulation for the thick Reissner's plate bending.

This paper presents the derivation and verifications of new boundary element formulation for plate bending problems. Unlike the conventional formulations, the proposed formulation is based on generalized variational principle. The Reissner's plate bending model is employed. It is considered a boundary element model because the final integral equation involves some boundary integrals that require a boundary discretization evaluation in order to be evaluated. Furthermore, all the unknowns are boundary variables. The model is completely new. It differs from the classical boundary element formulation in the way it is generated and consequently in the final equations. A generalized variational principle is used as a basis for its derivation, whereas the conventional boundary element formulation is based on Green's formula.

2. THE PROPOSED BEM HYBRID DISPLACEMENT FORMULATION

The energy functional for the Reissner's plate bending problems could be obtained as follows (Dym and Shamed [8]):

$$\begin{aligned}\Pi_3\big(\mathrm{u}_i(\mathbf{y}),\tilde{\mathrm{u}}_i(\mathbf{x}),\tilde{\mathrm{p}}_i(\mathbf{x})\big) &= \int_{\Omega(\mathbf{y})} \frac{1}{2}\left(\mathrm{M}_{\alpha\beta}(\mathbf{y})\chi_{\alpha\beta}(\mathbf{y}) + \mathrm{Q}_{3\alpha}(\mathbf{y})\psi_{3\alpha}(\mathbf{y})\right)d\Omega(\mathbf{y}) \\ &- \int_{\Omega(\mathbf{y})} \mathrm{b}_i(\mathbf{y})\mathrm{u}_i(\mathbf{y})\,d\Omega(\mathbf{y}) - \int_{\Gamma_P(\mathbf{x})} \bar{\mathrm{p}}_i(\mathbf{x})\tilde{\mathrm{u}}_i(\mathbf{x})\,d\Gamma(\mathbf{x}) \\ &+ \int_{\Gamma(\mathbf{x})} \tilde{\mathrm{p}}_i(\mathbf{x})\big(\tilde{\mathrm{u}}_i(\mathbf{x}) - \mathrm{u}_i(\mathbf{x})\big)\,d\Gamma(\mathbf{x})\end{aligned} \tag{1}$$

After integrating by parts the first two domain integral on the right hand side of equation (1) could be converted into boundary integrals (Rashed and Brebbia [9]) and taking into consideration the symmetry of the moment stress-resultant tensor and regrouping, equation (1) can be written as:

$$\begin{aligned}\Pi_3\big(u_i(\mathbf{y}), \tilde{u}_i(\mathbf{x}), \tilde{p}_i(\mathbf{x})\big) &= \int_{\Gamma(\mathbf{y})} \frac{1}{2} u_i(\mathbf{y}) p_i(\mathbf{y})\, d\Gamma(\mathbf{y}) - \int_{(\mathbf{y})} \frac{1}{2} M_{\alpha\beta,\beta}(\mathbf{y}) u_\alpha(\mathbf{y}) d\Omega(\mathbf{y}) \\ &- \int_{\Omega(\mathbf{y})} \frac{1}{2} Q_{3\alpha,\alpha}(\mathbf{y}) u_3(\mathbf{y}) d\Omega(\mathbf{y}) - \int_{\Omega(\mathbf{y})} b_i(\mathbf{y}) u_i(\mathbf{y}) d\Omega(\mathbf{y}) \\ &- \int_{\Gamma_P(\mathbf{x})} \tilde{u}_i(\mathbf{x}) \bar{p}_i(\mathbf{x})\, d\Gamma(\mathbf{x}) + \int_{\Gamma(\mathbf{x})} \tilde{p}_i(\mathbf{x}) \tilde{u}_i(\mathbf{x})\, d\Gamma(\mathbf{x}) \\ &- \int_{\Gamma(x,y)} \tilde{p}_i(\mathbf{x})\, u_i(\mathbf{y}) d\Gamma(\mathbf{x}, \mathbf{y}) \end{aligned} \tag{2}$$

The first four integrals in equation (2) involve the domain variables $u_i(\mathbf{x}), p_i(\mathbf{x})$. The following two integrals, involve the boundary variables $\tilde{p}_i(\mathbf{x}), \tilde{u}_i(\mathbf{x})$and the last integral involves both the domain and boundary variables.

A new variational boundary element formulation for the Ressiner plate bending model is obtained by representing the three independent field variables $u_i, \tilde{p}_i, \tilde{u}_i$ via approximate schemes. Hence variational principles are used to minimize the functional Π_3. The stationary condition (that corresponds to the equilibrium condition) for such a functional represents an approximate integral equation of the problems to approximate the domain terms $(u_i(\mathbf{y}), p_i(\mathbf{y})$ in which $\mathbf{y} \in \Omega(\mathbf{y}))$ in the first four integrals in equation (25). As in the indirect boundary element or the super-position formulation for Ressiner's plate bending problems (Mohareb and Rashed[10]),the rotation and the displacement components vector at any point ($\mathbf{y}$) inside the domain Ω could be approximated via a collection series. This series contains the product of fundamental solution ($U^*_{ki}(\mathbf{y}, \boldsymbol{\xi}_n)$) and an unknown set of fictitious concentrated tractions ($\gamma_k(\boldsymbol{\xi}_n)$) located at a set of arbitrary source points ($\boldsymbol{\xi}_n$), as follows:

$$u_i(\mathbf{y}) = U^*_{ki}(\mathbf{y}, \boldsymbol{\xi}_n)\gamma_k(\boldsymbol{\xi}_n) \tag{3}$$

Where the subscript (n) denotes arbitrary set of source points (its number could be taken later as the number of boundary nodes N) in which the fictitious tractions are applied along the direction (x_k).
In a similar way, the traction components at any point ($\mathbf{y}$) inside the domain Ω could be approximated via a collection series containing the products of fundamental solution ($P^*_{kl}(\mathbf{y}, \boldsymbol{\xi}_n)$) and the same unknown fictitious concentrated tractions ($\gamma_k(\boldsymbol{\xi}_n)$) which are located at the same set of points (ξ_n), as follows:

$$p_i(\mathbf{y}) = P^*_{ki}(\mathbf{y}, \xi_n)\gamma_k(\boldsymbol{\xi}_n) \tag{4}$$

Using the representation given in equations (3) and (4), the first integral on the right hand side of equation (2) could be re-written as follows:

$$\int_{\Gamma(\mathbf{y})} \frac{1}{2} u_i(\mathbf{y}) p_i(\mathbf{y})\, d\Gamma(\mathbf{y}) = \frac{1}{2} \gamma_k(\boldsymbol{\xi}_n) \left[\int_{\Gamma(\mathbf{y})} U^*_{ki}(\mathbf{y}, \boldsymbol{\xi}_n) P^*_{mi}(\mathbf{y}, \boldsymbol{\xi}_n) d\Gamma(y) \right] \gamma_m(\boldsymbol{\xi}_n) = \frac{1}{2} \{\boldsymbol{\gamma}\}^T_{1\times 3N} [\mathbf{F}]_{3N\times 3N} \{\boldsymbol{\gamma}\}_{3N\times 1} \tag{5}$$

where,

$$[\mathbf{F}]_{3N\times 3N} = \int_{\Gamma(\mathbf{y})} U^*_{ki}(\mathbf{y}, \boldsymbol{\xi}_n) P^*_{mi}(\mathbf{y}, \boldsymbol{\xi}_n) d\Gamma(\mathbf{y}) \tag{6}$$

In which (N) is the number of boundary points.
The second and third domain integrals on the right hand side of the equation (2) are set to zeros. This is done by making use of considered approximations in equations (3, 4) and placing the source points ($\boldsymbol{\xi}_n$)outside the plate boundary therefore:

$$\int_{\Omega(\mathbf{y})} \frac{1}{2} M_{\alpha\beta,\beta}(\mathbf{y}) u_\alpha(\mathbf{y}) d\Omega(\mathbf{y}) = 0 \tag{7}$$

and

$$\int_{\Omega(\mathbf{y})} \frac{1}{2} Q_{3\alpha,\alpha}(\mathbf{y}) u_3(\mathbf{y}) d\Omega(\mathbf{y}) = 0 \tag{8}$$

The last domain integral in equation (2) could be represented as follows:

$$\int_{\Omega(\mathbf{x})} b_i(\mathbf{y}) u_i(\mathbf{y}) d\Omega(\mathbf{y}) = \gamma_k(\boldsymbol{\xi}_n) \left[\int_{\Omega(\mathbf{y})} U^*_{ki}(\mathbf{y}, \boldsymbol{\xi}_n) b_i(\mathbf{y}) d\Omega(\mathbf{y}) \right] = \{\boldsymbol{\gamma}\}^T_{1\times 3N} \{B\}_{3N\times 1} \tag{9}$$

where

$$\{\mathbf{B}\}_{3N\times 1} = \int_{\Omega(\mathbf{y})} U^*_{ki}(\mathbf{y}, \xi_n) b_i(\mathbf{y}) d\Omega(\mathbf{y}) \tag{10}$$

It has to be noted that the vector $\{\mathbf{B}\}$ in equation (10) is similar to the one that appears in the classical direct boundary element method (Brebbia*et al.* [12]) and could be transformed to the boundary using similar ways as those given by Rashed and Brebbia [9].

The boundary displacement and traction vectors denoted by ($\tilde{u}_i$) and ($\tilde{p}_i$) are approximated using quadratic boundary elements therefore:

$$\tilde{u}_i(\boldsymbol{\eta}) = \sum_{j=1}^{j=3} \phi^j(\boldsymbol{\eta}) \;\; u_i^j(\mathbf{x_e}) \quad \forall\, \mathbf{x} \text{ in } \Gamma_e \tag{11}$$

$$\tilde{p}_i(\boldsymbol{\eta}) = \sum_{j=1}^{j=3} \phi^j(\boldsymbol{\eta}) \;\; p_i^j(\mathbf{x_e}) \quad \forall\, \mathbf{x} \text{ in } \Gamma_e \tag{12}$$

Where, $u_i^j(x_e)$ and $p_i^j(x_e)$ are vectors whose components are nodal(x_e) values for boundary displacements and boundary tractions.
Using the representation given in equations (11) and (12), the fifth integral of equation (2) could be re-written as follows:

$$\int_{\Gamma_P(\mathbf{x})} \tilde{u}_i(\mathbf{x})\bar{p}_i(\mathbf{x})\, d\Gamma(\mathbf{x}) = \sum_{\text{elements } (\Gamma_e)} \left\{u_i^j(\mathbf{x_e})\right\}^T \int_{\Gamma_e(\mathbf{x_e})} \phi^j(\boldsymbol{\eta})\bar{p}_i(\mathbf{x_e})\, d\Gamma(\mathbf{x_e}) \tag{13}$$

$$= \{\mathbf{u}\}_{1\times 3N}^T\{\bar{\mathbf{P}}\}_{3N\times 1} \tag{14}$$

where

$$\{\bar{\mathbf{P}}\}_{3N\times 1} = \int_{\Gamma_e(\mathbf{x_e})} \phi^j(\boldsymbol{\eta})\bar{p}_i(\mathbf{x_e})\, d\Gamma(\mathbf{x_e}) \tag{15}$$

In which (N) is the number of the used boundary elements nodes. The sixth integral of equation (2) could be re-written as follows:

$$\int_{\Gamma_P(\mathbf{x})} \tilde{p}_i(\mathbf{x})\tilde{u}_i(\mathbf{x})\, d\Gamma(\mathbf{x})$$

$$= \sum_{\text{elements } (\Gamma_e)} \left\{p_i^j(\mathbf{x_e})\right\}^T \left[\int_{\Gamma_e(\mathbf{x_e})} [\psi^j(\boldsymbol{\eta})][\phi^j(\boldsymbol{\eta})]^T d\Gamma(\mathbf{x_e})\right] u_i(\mathbf{x_e}) \tag{16}$$

$$= \{\mathbf{p}\}_{1\times 3N}^T[\mathbf{L}]_{3N\times 3N}\{\mathbf{u}\}_{3N\times 1} \tag{17}$$

Where

$$[\mathbf{L}]_{3N\times 3N} = \int_{\Gamma_e(\mathbf{x_e})} [\psi^j(\boldsymbol{\eta})][\phi^j(\boldsymbol{\eta})]^T d\Gamma(\mathbf{x_e}) \tag{18}$$

The last integral of equation (2) could be approximated as follows:

$$\int_{\Gamma(\mathbf{x})} \tilde{p}_i(\mathbf{x})\, u_i(\mathbf{x})\, d\Gamma(\mathbf{x})$$

$$= \sum_{\text{elements } (\Gamma_e)} \left\{p_i^j(\mathbf{x_e})\right\}^T \left[\int_{\Gamma_e(\mathbf{x_e})} \psi^j(\boldsymbol{\eta})U_{ki}^*(\mathbf{x_e}, \boldsymbol{\xi}_n)\, d\Gamma(\mathbf{x_e})\right] \gamma_k(\boldsymbol{\xi}_n) \tag{19}$$

$$= \{\mathbf{p}\}_{1\times 3N}^T[\mathbf{G}]_{3N\times 3N}^T\{\boldsymbol{\gamma}\}_{3N\times 1} \tag{20}$$

Where

$$[\mathbf{G}]_{3N\times 3N}^T = \int_{\Gamma_e(\mathbf{x_e})} \psi^j(\boldsymbol{\eta})U_{ki}^*(\mathbf{x_e}, \boldsymbol{\xi}_n) d\Gamma(\mathbf{x_e}) \tag{21}$$

It has to be noted that the matrix $[\mathbf{G}]$ in equation (21) is similar to the one that appears in the classical direct boundary element method (Karam and Telles [12]).

Using the approximations in equations (5, 7, 8, 9, 14, 17 and 20), equation (2) could be re-written as follows:

$$\Pi_3 = \frac{1}{2}\{\boldsymbol{\gamma}\}^T[\mathbf{F}]\{\boldsymbol{\gamma}\} - \{\mathbf{u}\}^T\{\bar{\mathbf{P}}\} + \{\mathbf{p}\}^T[\mathbf{L}]\{\mathbf{u}\} - \{\mathbf{p}\}^T[\mathbf{G}]^T\{\boldsymbol{\gamma}\} - \{\boldsymbol{\gamma}\}^T\{\mathbf{B}\} \tag{22}$$

The final system of algebraic equations could be obtained by computing the stationary conditions associate with$\Pi_3$in equation (22). This can be obtained by taking the first variation of equation (22) as follows:

$$\delta\Pi_3 = \{\delta\boldsymbol{\gamma}\}^{\mathrm{T}}([\mathbf{F}]\{\boldsymbol{\gamma}\} - [\mathbf{G}]\{\mathbf{p}\} - \{\mathbf{B}\}) + \{\delta\mathbf{u}\}^{\mathrm{T}}([\mathbf{L}]^{\mathrm{T}}\{\mathbf{p}\} - \{\overline{\mathbf{P}}\}) + \{\delta\mathbf{p}\}^{\mathrm{T}}([\mathbf{L}]\{\mathbf{u}\} - [\mathbf{G}]^{\mathrm{T}}\{\boldsymbol{\gamma}\}) \tag{23}$$

The functional $\Pi_3$is stationary when its first variation $\delta\Pi_3$vanishes for any arbitrary values of ($\delta\gamma(\boldsymbol{\xi}_n)$, $\delta u(\mathbf{x})$ and $\delta p(\mathbf{x})$).Therefore the corresponding generalized Euler`s equations are

$$[\mathbf{F}]\{\boldsymbol{\gamma}\} - [\mathbf{G}]\{\mathbf{P}\} - \{\mathbf{B}\} = 0 \tag{24}$$

$$[\mathbf{L}]^{\mathrm{T}}\{\mathbf{p}\} - \{\overline{\mathbf{P}}\} = 0 \tag{25}$$

$$[\mathbf{L}]\{\mathbf{u}\} - [\mathbf{G}]^{\mathrm{T}}\{\boldsymbol{\gamma}\} = 0 \tag{26}$$

The unknown vectors$\{\boldsymbol{\gamma}\}$ and $\{\mathbf{P}\}$are expressed in terms of the vector$\{\mathbf{P}\}$ to obtain a finalsystem of equations involving only the boundary unknown vector$\{\mathbf{P}\}$. Provided that the matrix $[\mathbf{G}]$is not singular (Karam and Telles [12]), equation (26) could be re-written as follows:

$$\{\boldsymbol{\gamma}\} = [[\,\mathbf{G}]^{\mathrm{T}}]^{-1}[\mathbf{L}]\{\mathbf{u}\} \tag{27}$$

Substituting equation (27) into equation (24) gives:

$$\{\mathbf{P}\} = [\mathbf{G}]^{-1}[\mathbf{F}][[\,\mathbf{G}]^{\mathrm{T}}]^{-1}[\mathbf{L}]\{\mathbf{u}\} - [\mathbf{G}]^{-1}\{\mathbf{B}\} \tag{28}$$

Substituting equation (28) into equation (25) gives:

$$[\mathbf{L}]^{\mathrm{T}}[\mathbf{G}]^{-1}[\mathbf{F}][[\,\mathbf{G}]^{\mathrm{T}}]^{-1}[\mathbf{L}]\{\mathbf{u}\} - [\mathbf{L}]^{\mathrm{T}}[\mathbf{G}]^{-1}\{\mathbf{B}\} - \{\overline{\mathbf{P}}\} = 0 \tag{29}$$

Introducing the following definitions:

$$[\mathbf{R}] = [[\,\mathbf{G}]^{\mathrm{T}}]^{-1}[\mathbf{L}] \tag{30}$$

Hence equation (29) could be re-written as follows:

$$[\mathbf{R}]^{\mathrm{T}}[\mathbf{F}][\mathbf{R}]\{\mathbf{u}\} - [\mathbf{R}]^{\mathrm{T}}\{\mathbf{B}\} - \{\overline{\mathbf{P}}\} = 0 \tag{31}$$

Defining:

$$[\mathbf{K}] = [\mathbf{R}]^{\mathrm{T}}[\mathbf{F}][\mathbf{R}] \tag{32}$$

and

$$\{\mathbf{Q}\} = [\mathbf{R}]^{\mathrm{T}}\{\mathbf{B}\} + \{\overline{\mathbf{P}}\} \tag{33}$$

Hence, equation (29) could be re-written as follows:

$$[\mathbf{K}]_{3N\times3N}\{\mathbf{u}\}_{3N\times1} = \{\mathbf{Q}\}_{3N\times1} \tag{34}$$

It has been noted that the obtained $[\mathbf{K}]$ or the stiffness matrix is symmetric, positive definite and similar to the one obtained from the finite element method (Zienkiewicz [13]). The vectors $\{\mathbf{u}\}$ and $\{\mathbf{Q}\}$ are the corresponding vectors of boundary displacements and forces.

3. SOLUTION AT INTERNAL POINTS

After solving equation (34), the vector $\{\boldsymbol{\gamma}\}$ is computed from equation (27). Hence the internal displacement vector at any point ($\mathbf{y}$) inside the domain $(\Omega)$is computed using equation (3) as follows: follows:

$$u_{\alpha}(\mathbf{y}) = U^{*}_{k\alpha}(\mathbf{y}, \boldsymbol{\xi}_n)\gamma_k(\boldsymbol{\xi}_n) \tag{35}$$

$$u_{3}(\mathbf{y}) = U^{*}_{k3}(\mathbf{y}, \boldsymbol{\xi}_n)\gamma_k(\boldsymbol{\xi}_n) \tag{36}$$

Stress resultants at any point ($\mathbf{y}$) inside the domain (Ω) are computed after carrying out relevant derivatives as follows:

$$u_{\alpha,\gamma}(\mathbf{y}) = U^{*}_{k\alpha,\gamma}(\mathbf{y}, \boldsymbol{\xi}_n)\gamma_k(\boldsymbol{\xi}_n) \tag{37}$$

and

$$u_{3,\gamma}(\mathbf{y}) = U^{*}_{k3,\gamma}(\mathbf{y}, \boldsymbol{\xi}_n)\gamma_k(\boldsymbol{\xi}_n) \tag{38}$$

Expanding the index (k) to (β) and (3) gives:

$$u_{\alpha,\gamma}(\mathbf{y}) = U^{*}_{\beta\alpha,\gamma}(\mathbf{y}, \boldsymbol{\xi}_n)\gamma_\beta(\boldsymbol{\xi}_n) + U^{*}_{3\alpha,\gamma}(\mathbf{y}, \boldsymbol{\xi}_n)\gamma_3(\boldsymbol{\xi}_n) \tag{39}$$

and

$$u_{3,\gamma}(\mathbf{y}) = U^{*}_{\beta3,\gamma}(\mathbf{y}, \boldsymbol{\xi}_n)\gamma_\beta(\boldsymbol{\xi}_n) + U^{*}_{33,\gamma}(\mathbf{y}, \boldsymbol{\xi}_n)\gamma_3(\boldsymbol{\xi}_n) \tag{40}$$

The new derivatives $U^{*}_{\alpha\beta,\gamma}, U^{*}_{3\alpha,\gamma}, U^{*}_{\beta3,\gamma}, U^{*}_{33,\gamma}$ are given in the appendix. It has to be noted that unlike the direct boundary element method (Vander Weeën [1]), all relevant derivatives herein are carried out with respect to the coordinate of the field point ($x_{\gamma}(\mathbf{y})$).

4. NUMERICAL EXAMPLES

4.1 Clamped circular plate subject to domain load

A thin, circular plate (radius a, thickness t, Young's modulus E) is clamped along its outer boundary as shown in Figure 1 and is subject to a uniformly distributed load with intensity $P=P_0$. The results for the generalized displacements at points (A and B) are evaluated by using different meshes and presented in table (1) together with analytical values and results obtained from the convention direct boundary elements using quadratic elements is given by Rashed [14].

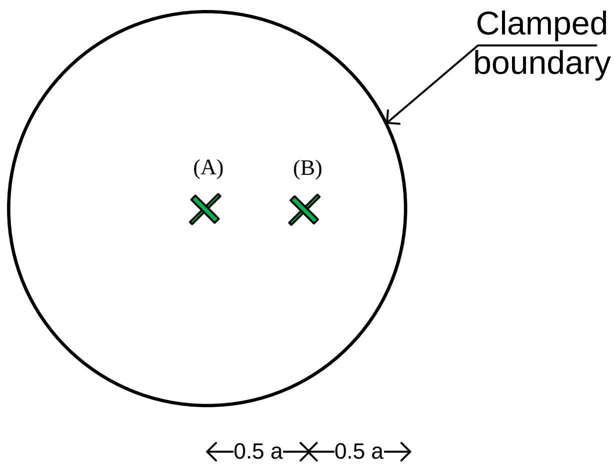

Figure 1: Clamped circular plate subjected to domain load

	$u_3(A) \times \frac{128D}{qa^4}$	$u_3(B) \times \frac{128D}{qa^3}$
Analytical solution	2	1.125
Presented BEM	2.0358	1.1356
Conv. direct. BEM [Rashed (2000)]	2.0307	1.1539
Finite element method	2.0375	1.1356

Table 1: Results for the generalized displacements at point (A, B)

It can be seen from Tables (1) that results for the present variational formulation and conventional boundary element method is accurate with respect to the analytical values.

4.2 Square plate with a square central opening

A square plate with a square central opening is subjected to a uniform surface load p. The external edges of the plate are simply supported and the internal edges are free, as shown in Figure 2-a) the deflection at the points A, B, and C are calculated. The results are evaluated by using mesh indicated in Figure 2-b) and presented in Table (2) together with results obtained from the conventional direct boundary element method, finite element method and the Finite difference method which given by Tottenham [15], Assume Poisson's ratio ν equal to 1/6.

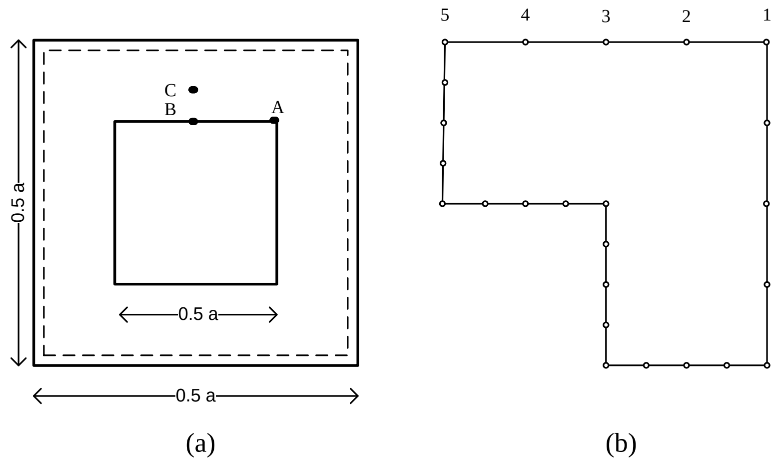

Figure 2 Square plate with central opening subjected to domain load

	Displacement (x pa/100D)				
	Boundary element method				
Point	Presented BEM	Indirect	Direct	Finite element method	Finite difference method
A	0.21802	0.2188	0.2191	0.2185	0.2174
B	0.2807	0.3107	0.2818	0.3156	0.3006
C	0.1535	0.1558	0.1559	-	0.1541

Table 2: Results for the generalized displacements at point (A, B, C)

It can be seen from Tables (2) that results for for the present variational formulation, conventional boundary element method, finite element method and finite difference method are in good agreement.

4.3 Curved Plate Bridge

The 0.3 m curved Plate Bridge shown Figure is subjected to a uniform surface load of intensity -2.0 t/m^2.The Young's modulus for the plate material is E=2.5×10^7 t/m^2 and Poisson's ratio ν equal to 0.25. Figures (4, 5) demonstrate the Deflection and Moment Mxx at center line of plate are plotted from CBEM and present BEM.

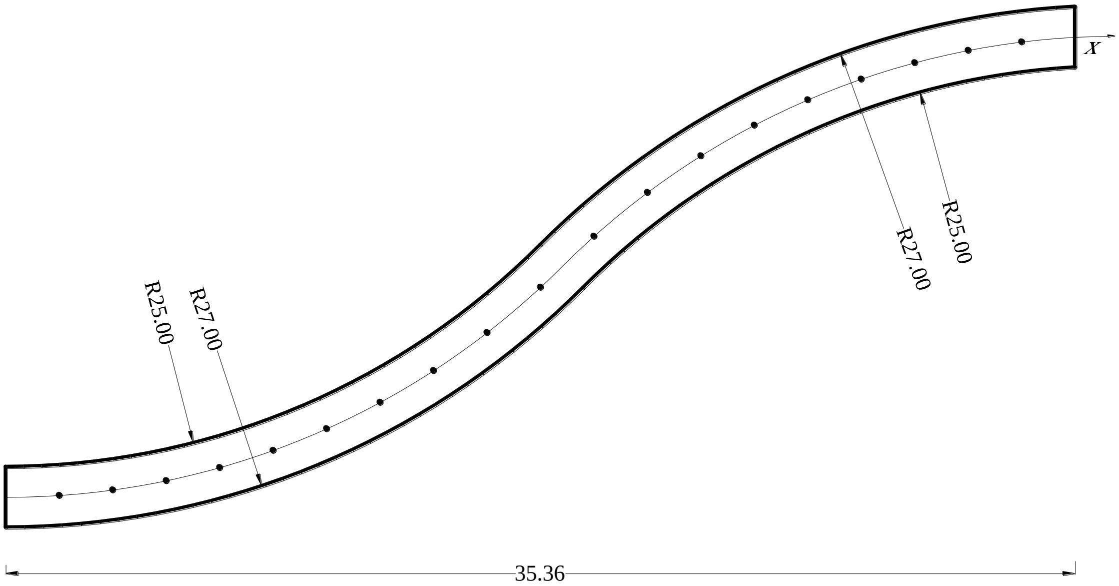

Figure 3 Curved Plate Bridge subjected to domain load

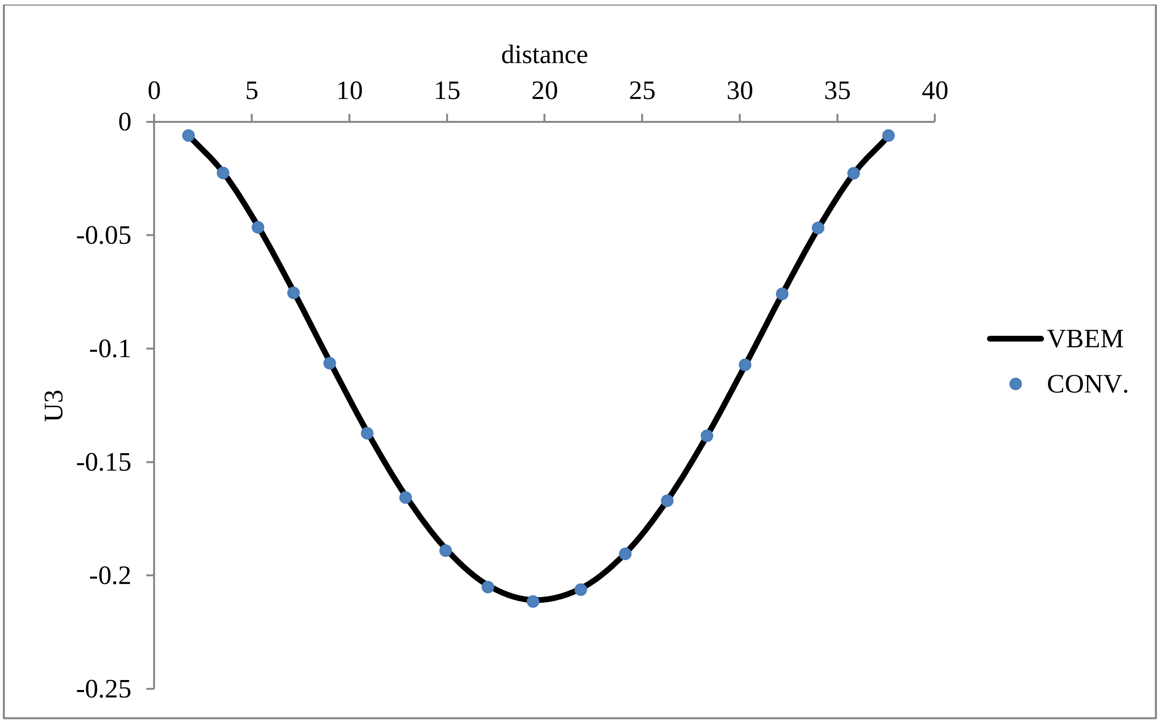

Figure 4: Deflection along center line

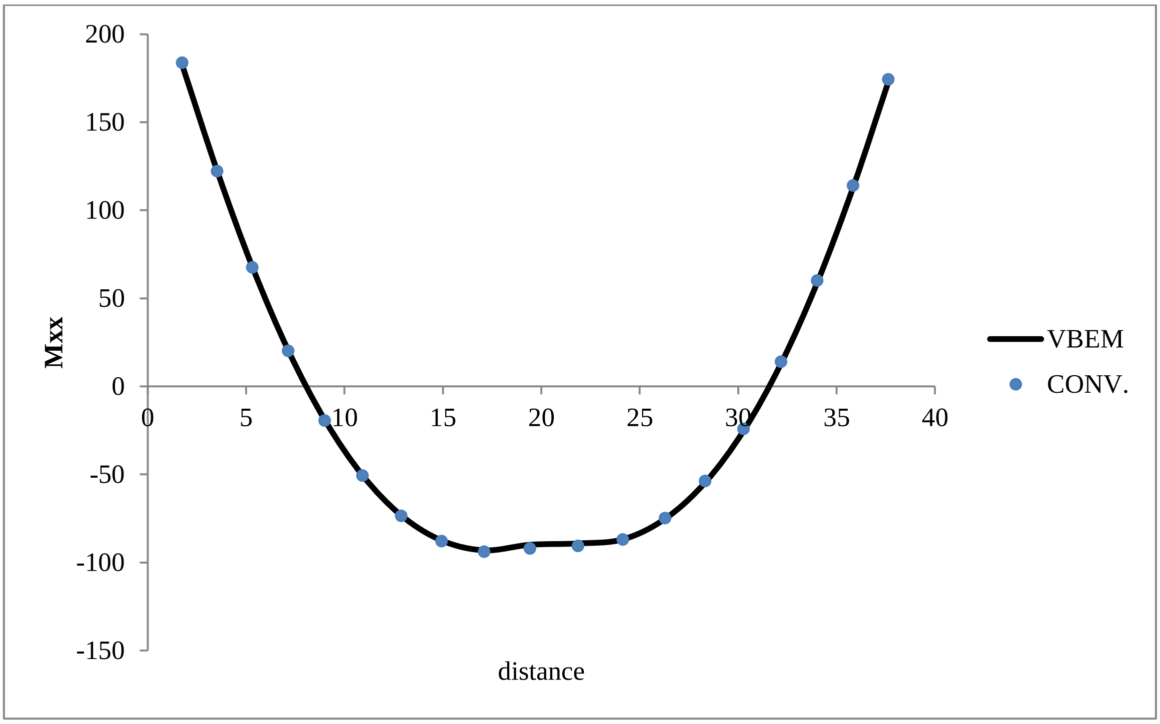

Figure 5: Moment M_{xx} along center line

5. CONCLUSIONS

In this paper, a variational boundary element formulation of Reissner's plate bending problems was derived. The formulation was based on minimizing the relevant energy functional.A collocation based series is used to remove domain integrals. Hence a fully boundary integral equation is formulated. The formulation was transformed into matrix equations using quadratic boundary elements, and was implemented into a computer code. Several examples with different boundary conditions were tested. It was demonstrated that the present formulation results were more accurate compared to results obtained from the conventional direct boundary elements, even with fewer number of elements. In addition, the present formulation produces symmetric stiffness matrix similar to that obtained from the finite element method. Therefore such formulation is very suitable to be coupled with boundary and finite elements or to produce a new family of super finite elements; which will be considered in future research.

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